HW IV A 第 題
(答題不得寫在紅綫外) 第 頁

Let $(x_n)$ be a bounded seq, $y_n := \sup\{x_n, x_{n+1}, \cdots\}$

$v \in \mathbb{R}$ is said to be an essential upper bound of $(x_n)$ if $\exists N \in \mathbb{N}$ such that $x_n \leq v \ \forall n \geq N$.

Let

$$V := \{v \in \mathbb{R} : v \text{ is an essential upper bound of } (x_n)\}.$$

and

$$L := \{\ell \in \mathbb{R} : \exists \text{ a subseq. of } (x_n) \text{ convergent to } \ell\}.$$

Show that

Q1 By what theorem (how it is stated), you can conclude that $y^* := \lim_n y_n$ does exist and $y^* = \inf\{y_n : n \in \mathbb{N}\}$?

Q2 Let $\alpha \in \mathbb{R}$. Then

$$y^* \leq \alpha \text{ iff } \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } x_n < \alpha + \varepsilon \ \forall n \geq N$$

and

$$\alpha \leq y^* \text{ iff } \forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N \text{ s.t. } \alpha - \varepsilon < x_n.$$

Q3 What are $y_n$, $y^*$, $V$ and $L$ if $x_n = 1/n \ \forall n$ (do the same for $x_n = 1 - \frac{1}{n} \ \forall n$).

Q4 Show that any u.b. of $(x_n)$ is an essential u.b of $(x_n)$, and that any l.b. of $(x_n)$ is a l.b. of $V$ so $\inf V$ exists in $\mathbb{R}$.

Q5 $\inf V = \max L = y^*$ (denoted by $\limsup_n x_n$).